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# On an Open Problem of P. Turán Concerning Birkhoff Interpolation Based on the Roots of Unity

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In 1974 P. Turán (*J. Approx. Theory* **29** (1980), 23–85) raised many interesting open problems in approximation theory, some of which are on Birkhoff interpolation. The object of this paper is to answer Problem XLVI in the affirmative. In fact in our two main theorems we treat a case that is more general than the case of the problem. 

## INTRODUCTION

Following G. D. Birkhoff (see [3]), P. Turán and his associates have initiated the problem of (0, 2) interpolation where the value and second derivative of the interpolatory polynomial are prescribed at given points ("knots"). This problem has been extensively discussed in Chapter 12 of [3] for the case that the knots are the zeros of $(1-x^2)\,P'_{n-1}(x)$, $P_n$ being the $n$th-degree Legendre polynomial.

The problem of Birkhoff interpolation on the unit circle has been initiated by O. Kis [2] for the special knots $Z_n=\{z_{kn}=e^{2\pi ik/n};\ k=1,2,\ldots,n\}$, the $n$th roots of unity. The early results dealt with the problems of existence, uniqueness, and explicit representation and the problem of convergence of (0, 2) interpolation based on the nodes $Z_n$. In the same paper O. Kis also resolved the corresponding problem of (0, 1, 3) interpolation. Later A. Sharma [6] considered the more general case of



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$(0, m)$ interpolation based on the nodes $Z_n$. The main results of Sharma and Kis may be described by the following theorems.

THEOREM A. *If* $z_{kn} = \exp(2\pi ki/n)$, $k = 1, 2, \ldots, n$, *then the unique polynomial* $R_{n,m}(z)$ *satisfying*

$$R_{n,m}(z_{\nu n}) = \alpha_{\nu n}, \qquad R_{n,m}^{(m)}(z_{\nu n}) = \beta_{\nu n}, \qquad \nu = 1, 2, \ldots, n \geqslant m, \tag{1.1}$$

*is*

$$R_{n,m}(z) = \sum_{k=1}^{n} \alpha_{kn} A_{k,n,m}(z) + \sum_{k=1}^{n} \beta_{kn} B_{k,n,m}(z), \tag{1.2}$$

*where the fundamental polynomials* $A_{k,n,m}(z)$, $B_{k,n,m}(z)$ *are given by*

$$A_{k,n,m}(z) = l_{kn}(z) - \frac{1}{n} \sum_{\nu=0}^{n-1} \lambda_{\nu,n,m} z_{kn}^{n-\nu}(z^{n+\nu} - z^{\nu}), \tag{1.3}$$

$$B_{k,n,m}(z) = \frac{z^n - 1}{n} \sum_{\nu=0}^{n-1} z_{kn}^{m-\nu} \mu_{\nu,n,m} z^{\nu} \tag{1.4}$$

*with*

$$l_{kn}(z) = \frac{z_{kn}}{n} \frac{z^n - 1}{z - z_{kn}}, \qquad k = 1, 2, \ldots, n, \tag{1.5}$$

$$\lambda_{\nu,n,m} = \frac{(\nu)_m}{(\nu+n)_m - (\nu)_m}, \qquad \mu_{\nu,n,m} = \frac{1}{(\nu+n)_m - (\nu)_m}, \tag{1.6}$$

$$(\nu)_m = \nu(\nu-1) \cdots (\nu - m + 1). \tag{1.7}$$

THEOREM B. *Let* $f(z)$ *be analytic in* $|z| < 1$ *and continuous in* $|z| \leqslant 1$. (*This class of functions will be denoted by* $C[|z| \leqslant 1]$.) *Let* $w(f, \delta)$ *be the modulus of continuity of* $f(\exp i\theta)$, $0 \leqslant \theta \leqslant 2\pi$. *If*

$$\lim_{\delta \to 0} w(f, \delta) \log \delta = 0 \tag{1.8}$$

*and*

$$\beta_{kn} = o\left(\frac{n^m}{\log n}\right), \qquad k = 1, 2, \ldots, n, \tag{1.9}$$

*then the polynomial*

$$R_{n,m}(f, Z) = \sum_{k=1}^{n} f(z_{kn}) A_{k,n,m}(z) + \sum_{k=1}^{n} \beta_{k,n,m} B_{k,n,m}(z) \tag{1.10}$$

*converges uniformly to* $f(z)$ *in* $|z| \leqslant 1$.

In 1974, J. Szabados [8] considered the interpolatory procedure $R_{n,m}(f, z)$ in the special case $m=2$ and $\beta_{kn}=0$, $k=1, 2,..., n$. We shall denote this special case by

$$I_{n,2}(f, z)=\sum_{k=1}^{n} f(z_{kn})\, A_{k,n,2}(z). \tag{1.11}$$

Let $w(h)$ be an arbitrary modulus of continuity, and denote by $AC(w)$ that class of functions $f(z)$ which are analytic in $|z|<1$, and whose modulus of continuity $w(f, h)$ in $|z|\leqslant 1$ is of order $O(w(h))$ (as $h\to +0$). How does $I_n(f, z)$ approximate a function $f(z)\in AC(w)$? The answer to this problem is given by

THEOREM C (J. Szabados). *If $f(z)\in AC(w)$ then*

$$\max_{|z|=1} |f(z)-I_{n,2}(f, z)| = O(w(1/n)\log n). \tag{1.12}$$

*Moreover, the above estimate* (1.12) *cannot be improved; viz., to every $w(h)$, there exists a function $f(z)\in AC(w)$ such that*

$$\limsup_{n\to\infty} \frac{|f(1)-I_{n,2}(f, 1)|}{w(1/n)\log n} > 0. \tag{1.13}$$

In view of Theorem B, P. Turán [9] raised the following problem concerning $I_{n,2}(f, Z)$ as defined by (1.11).

PROBLEM XLVI. Is it true that, for all $f(z)$ analytic in $|z|<1$ and continuous in $|z|\leqslant 1$,

$$\int_{c:|z|=1} |f(z)-I_{n,2}(f, z)|^2\, |dz| = 0? \tag{1.14}$$

The object of this paper is to answer this problem, namely,

THEOREM 1. *Let $f(z)\in C[|z|\leqslant 1]$ and let*

$$I_{n,m}(f, z)=\sum_{k=1}^{n} f(z_{kn})\, A_{k,n,m}(z), \tag{1.15}$$

*where $A_{k,n,m}(z)$ is defined by* (1.3). *Then*

$$\left(\frac{1}{2\pi}\int_{C:|z|=1} |f(z)-I_{n,m}(f, z)|^2\, |dz|\right)^{1/2} \leqslant C_1 w\left(f, \frac{1}{n}\right), \tag{1.16}$$

*where $C_1$ is a positive absolute constant and $w(f, 1/n)$ has the same meaning as in Theorem B.*

THEOREM 2. *Let $f(z) \in C[|z| \leqslant 1]$ and $R_{n,m}(f, z)$ be as defined by* (1.10). *Then*

$$\lim_{n \to \infty} \left( \frac{1}{2\pi} \int_{C:|z|=1} |f(z) - R_{n,m}(f, z)|^2 \, |dz| \right)^{1/2} = 0$$

*provided $\beta_{kn} = o(n^m)$, $k = 1, 2, \ldots, n$.*

## 2. PRELIMINARIES

We need the well-known fact that the functions 1, $z$, $z^2$,... are mutually orthogonal on the unit circle $c$: $|z| = 1$; i.e., we have

$$\int_C z^k z^{-n} \, |dz| = \frac{1}{i} \int_C z^{k-n-1} \, dz = 0 \qquad \text{for} \quad k \neq n. \tag{2.1}$$

For our purpose we need another known representation of $l_{kn}(z)$. This is given by

$$l_{kn}(z) = \frac{1}{n} \sum_{v=0}^{n-1} z_{kn}^{n-v} z^v. \tag{2.2}$$

From (2.2) we may conclude that

$$l_{kn}(z_{jn}) = \frac{1}{n} \sum_{v=0}^{n-1} z_{kn}^{n-v} z_j^v = \frac{1}{n} \sum_{v=0}^{n-1} z_{kn}^{n-v} \bar{z}_{jn}^{n-v}. \tag{2.3}$$

Next, we set

$$a_{vn} = \frac{1}{n} \sum_{k=1}^{n} e_{kn} z_{kn}^{n-v}, \qquad v = 0, 1, \ldots, n-1, \tag{2.4}$$

where $e_{kn}$ are arbitrary numbers. Then ffrom (2.2), (2.3), and (2.4) we have

$$\sum_{v=0}^{n-1} |a_{vn}|^2 = \frac{1}{n} \sum_{k=1}^{n} |e_{kn}|^2. \tag{2.5}$$

Proof of (2.5) can be given as follows:

$$\sum_{\nu=0}^{n-1} |a_{\nu n}|^2 = \sum_{\nu=0}^{n-1} a_{\nu n}\bar{a}_{\nu n}$$
$$= \frac{1}{n^2} \sum_{\nu=0}^{n-1} \sum_{k=1}^{n} e_{kn} z_{kn}^{n-\nu} \sum_{j=1}^{n} \bar{e}_{jn} \bar{z}_{jn}^{n-\nu}$$
$$= \frac{1}{n} \sum_{k=1}^{n} \sum_{j=1}^{n} e_{kn}\bar{e}_{jn} \frac{1}{n} \sum_{\nu=0}^{n-1} z_{kn}^{n-\nu} \bar{z}_{jn}^{n-\nu}$$
$$= \frac{1}{n} \sum_{k=1}^{n} \sum_{j=1}^{n} e_{kn}\bar{e}_{jn} l_{kn}(z_{jn}) = \frac{1}{n} \sum_{k=1}^{n} |e_{kn}|^2.$$

## 3. Some Lemmas

For the proof of our theorems we need the following lemmas.

Lemma 3.1. *Let $L_n(f, z)$ denote the Lagrange interpolation polynomial based on the zeros $z_{kn}$. Then*

$$\int_{C:|z|=1} |L_n(g, z)|^2 \, |dz| = \frac{2\pi}{n} \sum_{k=1}^{n} |g(z_{kn})|^2. \tag{3.1}$$

*Proof.* We know that

$$L_n(g, z) = \sum_{k=1}^{n} g(z_{kn}) \, l_{kn}(z)$$
$$= \sum_{k=1}^{n} g(z_{kn}) \frac{1}{n} \sum_{\nu=0}^{n-1} z_{kn}^{n-\nu} z^{\nu}$$
$$= \sum_{\nu=0}^{n-1} z^{\nu} \frac{1}{n} \sum_{k=1}^{n} g(z_{kn}) \, z_{kn}^{n-\nu}$$
$$= \sum_{\nu=0}^{n-1} z^{\nu} b_{\nu n}, \qquad b_{\nu n} = \frac{1}{n} \sum_{k=1}^{n} g(z_{kn}) \, z_{kn}^{n-\nu}.$$

Therefore, on using (2.1)–(2.4) we have

$$\int_{C:|z|=1} |L_n(g, z)|^2 \, |dz| = 2\pi \sum_{\nu=0}^{n-1} |b_{\nu n}|^2$$
$$= \frac{2\pi}{n} \sum_{k=1}^{n} |g(z_{kn})|^2$$

(here in applying (2.4) we put $e_{kn} = g(z_{kn})$).

This proves the lemma. The next lemma is due to O. Kis [2].

LEMMA 3.2. *Let $f(z) \in C[|z| \leqslant 1]$. Let $w(f, \delta)$ be the modulus of continuity of $f(\exp i\theta)$, $0 \leqslant \theta \leqslant 2\pi$. Then there exists a polynomial $F_n(z)$ of degree $\leqslant n-1$ such that for $|z| \leqslant 1$*

$$|f(z) - F_n(z)| \leqslant 6w(f, 1/n) \tag{3.2}$$

*and*

$$|F_n^{(m)}(z)| \leqslant C_2 n^m w(f, 1/n). \tag{3.3}$$

In the next lemma we state some facts about Lagrange interpolation polynomials.

LEMMA 3.3. *Let $f(z) \in C[|z| \leqslant 1]$ and denote by $P_{n-1}(z)$ the polynomial of best approximation to $f(z)$ on $|z| \leqslant 1$; viz.,*

$$|f(z) - P_{n-1}(z)| \leqslant E_{n-1}(f). \tag{3.4}$$

*Then we have*

$$\left(\frac{1}{2\pi}\int_{C:|z|=1} |L_n(f-P_{n-1}, z)|^2\, |dz|\right)^{1/2} \leqslant E_{n-1}(f), \tag{3.5}$$

$$\left(\frac{1}{2\pi}\int_{C:|z|=1} |L_n(f, z) - f(z)|^2\, |dz|\right)^{1/2} \leqslant C_2 E_{n-1}(f), \tag{3.6}$$

*and*

$$\left(\frac{1}{2\pi}\int_{C} |L_n(f-F_n, z)|^2\, |dz|\right)^{1/2} \leqslant C_3 w\left(f, \frac{1}{n}\right), \tag{3.7}$$

*where $F_n(z)$ are polynomials of degree $\leqslant n-1$ (as stated in Lemma 3.2).*

*Proof.* Proof of (3.5) is an immediate consequence of (3.1) and (3.4); proof of (3.6) follows from (3.5) and (3.4); proof of (3.7) is a consequence of (3.2) and (3.6). Next we shall prove

LEMMA 3.4. *Let $I_{n,m}(z)$ be as defined by (1.15). Let $F_n(z)$ be the polynomial of degree $\leqslant n-1$ satisfying (3.2) and (3.3). Also let $P_{n-1}(z)$ be the polynomial of best approximation to $f(z)$ satisfying (3.4). Then there exist absolute positive constants $C_4$ and $C_5$ independent of $n$ such that*

$$\left(\frac{1}{2\pi}\int_{C:|z|=1} |I_{n,m}(f-F_n, z)|^2\, |dz|\right)^{1/2} \leqslant C_4 w\left(f, \frac{1}{n}\right) \tag{3.8}$$

*and*

$$\left(\frac{1}{2\pi}\int_{C:|z|=1} |I_{n,m}(f-P_{n-1}, z)|^2\, |dz|\right)^{1/2} \leqslant C_5 E_{n-1}(f). \tag{3.9}$$

*Proof.* Here we will prove only (3.9) for the proof of (3.8) and (3.9) are the same. Let $g(z)=f(z)-P_{n-1}(z)\in C[|z|\leqslant 1]$. Then on using (1.15), (1.3), (1.5), (1.6), and (1.7) we have

$$\begin{aligned}I_{n,m}(g,z)&=\sum_{k=1}^{n} g(z_{kn})\, l_{kn}(z)\\&\quad-\frac{1}{n}\sum_{k=1}^{n} g(z_{kn})\sum_{v=0}^{n-1} z_{kn}^{n-v}\lambda_{v,n,m}(z^{v+n}-z^{v})\\&=L_n(g,z)-\sum_{v=0}^{n-1}\lambda_{v,n,m}(z^{v+n}-z^{v})\frac{1}{n}\sum_{k=1}^{n} g(z_{kn})\, z_{kn}^{n-v}\\&=L_n(g,z)-\sum_{v=0}^{n-1}\lambda_{v,n,m}(z^{v+n}-z^{v})\, a_{vn},\end{aligned}$$

where

$$a_{vn}=\frac{1}{n}\sum_{k=1}^{n} g(z_{kn})\, z_{kn}^{n-v}. \tag{3.10}$$

Next, on using (2.1) and (3.1) we obtain

$$\begin{aligned}&\int_{|z|=1} |I_{n,m}(g,z)|^2\,|dz|\\&\quad\leqslant 2\int_{|z|=1} |L_n(g,z)|^2\,|dz|\\&\qquad+2\int_{|z|=1}\left|\sum_{v=0}^{n-1}\lambda_{v,n,m}a_{vn}z^{v+n}\right|^2|dz|\\&\qquad+2\int_{|z|=1}\left|\sum_{v=0}^{n-1}\lambda_{v,n,m}a_{v,m}z^{v}\right|^2|dz|\\&\quad=\frac{4\pi}{n}\sum_{k=1}^{n}|g(z_{kn})|^2+8\pi\sum_{v=0}^{n-1}|\lambda_{v,n,m}|^2\,|a_{v,n}|^2.\end{aligned}$$

If is known that $\lambda_{v,n,m}$ as defined by (1.6) satisfy the inequality $(\lambda_{vnm})\leqslant C_6$, where $C_6$ is an absolute constant. Therefore

$$\begin{aligned}&\frac{1}{2\pi}\int_{|z|=1}|I_{n,m}(g,z)|^2\,|dz|\\&\quad\leqslant\frac{2}{n}\sum_{k=1}^{n}|f(z_{kn})-P_{n-1}(z_{kn})|^2+4C_6^2\sum_{v=0}^{n-1}|a_{vn}|^2.\end{aligned} \tag{3.11}$$

Now, on using (2.4), (2.5), and (3.10) we obtain

$$\sum_{\nu=0}^{n-1} |a_{\nu n}|^2 = \frac{1}{n} \sum_{k=1}^{n} (g(z_{kn}))^2$$
$$= \frac{1}{n} \sum_{k=1}^{n} |f(z_{kn}) - P_{n-1}(z_{kn})|^2 \tag{3.12}$$

on combining (3.11), (3.12) and making use of (3.4) we obtain

$$\left(\frac{1}{2\pi} \int_{|z|=1} |I_{n,m}(g, z)|^2 \, |dz|\right)$$
$$\leqslant \left(\frac{4C_6^2+2}{n}\right) \sum_{k=1}^{n} |f(z_{kn}) - P_{n-1}(z_{kn})|^2$$
$$\leqslant (2+4C_6^2)(E_{n-1}(f))^2.$$

From this we obtain (3.9). This proves the lemma. The next two lemmas are needed for the proof of our Theorem 2.

LEMMA 3.5. *Let $F_n(z)$ be the approximating polynomial of degree $\leqslant n-1$ satisfying* (3.2), (3.3) *and let $I_{n,m}(f, z)$ be as defined by* (1.15). *Then we have*

$$\left(\frac{1}{2\pi} \int_{|z|=1} |F_n(z) - I_{n,m}(F_n, z)|^2 \, |dz|\right)^{1/2} \leqslant C_6 w\left(f, \frac{1}{n}\right). \tag{3.13}$$

*Proof.* We know that

$$F_n(z) - I_{n,m}(F_n, z) = \sum_{k=1}^{n} F_n^{(m)}(z_{kn}) \, B_{k,n,m}(z), \tag{3.14}$$

where $B_{k,n,m}(z)$ is defined by (1.4). On substituting the value of $B_{k,n,m}(z)$ from (1.4) into (3.14) we obtain

$$F_n(z) - I_{n,m}(F_n, z)$$
$$= \sum_{k=1}^{n} F_n^{(m)}(z_{kn}) \frac{z^n - 1}{n} \sum_{\nu=0}^{n-1} z_{kn}^{m-\nu} z^\nu \mu_{\nu,n,m}$$
$$= \sum_{\nu=0}^{n-1} (z^{\nu+n} - z^\nu) \, \mu_{\nu,n,m} \frac{1}{n} \sum_{k=1}^{n} F_n^{(m)}(z_{kn}) \, z_{kn}^{m-\nu}$$
$$= \sum_{\nu=0}^{n-1} (z^{\nu+n} - z^\nu) \, \mu_{\nu,n,m} \, d_{\nu,n,m}, \tag{3.15}$$

where we set

$$d_{\nu,n,m} = \frac{1}{n} \sum_{k=1}^{n} F_n^{(m)}(z_{kn})\, z_{kn}^{m-\nu}. \tag{3.16}$$

We also need

$$|\mu_{\nu,n,m}| \leqslant C_7/n^m, \qquad \nu = 0, 1, ..., n-1, \tag{3.17}$$

which follows from (1.6). Now on using (2.1), (2.4), (2.5), (3.15), (3.16), (3.17), and (3.3) we have

$$\begin{aligned}
&\frac{1}{2\pi} \int_{|z|=1} |F_n(z) - I_{n,m}(F_n, z)|^2\, |dz| \\
&= 2 \sum_{\nu=0}^{n-1} |\mu_{\nu,n,m}|^2\, |d_{\nu,n,m}|^2 \\
&\leqslant \frac{2C_7^2}{n^{2m}} \sum_{\nu=0}^{n-1} |d_{\nu,n,m}|^2 = \frac{C_8}{n^{2m+1}} \sum_{k=1}^{n} |F_n^{(m)}(z_{kn})|^2 \\
&\leqslant C_9 \left( w\left(f, \frac{1}{n}\right)\right)^2,
\end{aligned}$$

from which Lemma 3.5 follows. Next we state

LEMMA 3.6. *We have*

$$\left( \frac{1}{2\pi} \int_{|z|=1} \left| \sum_{k=1}^{n} \beta_{kn} B_{k,n,m}(z) \right|^2 |dz| \right)^{1/2} = o(1) \tag{3.18}$$

*provided*

$$\beta_{kn} = o(n^m), \qquad K = 1, 2, ..., n, \tag{3.19}$$

*where* $B_{k,n,m}(z)$ *are defined by* (1.4). *Proof of this lemma is essentially the same as that of Lemma* 3.5, *so we omit the details.*

## 4. PROOF OF THEOREM 1

From (1.15) we have

$$\begin{aligned}
f(z) - I_{n,m}(f, z) = f(z) - F_n(z) + F_n(z) - I_{n,m}(F_n, z) \\
+ I_{n,m}(F_n - f, z).
\end{aligned} \tag{4.1}$$

Therefore on using (4.1), (3.2), (3.13), and (3.8)

$$\begin{aligned}\int_{|z|=1} |f(z)-I_{n,m}(f,z)|^2\,|dz| &\leqslant C_{10}\left\{\int_{|z|=1} |f(z)-F_n(z)|^2\,|dz|\right.\\ &\quad+\int_{|z|=1} |F_n(z)-I_{n,m}(F_n,z)|^2\,|dz|\\ &\quad\left.+\int_{|z|=1} |I_{n,m}(F_n-f,z)|^2\,|dz|\right\}\\ &\leqslant C_{11}\left(w\left(f,\frac{1}{n}\right)\right)^2.\end{aligned}$$

From the above Theorem 1 follows at once. Proof of Theorem 2 is a direct consequence of (1.16), (3.18). In a recent paper by Sharma and Vertesi [7], $L_p$ convergence of Lagrange interpolation polynomials has been settled. Similar questions can be asked for the operator $I_{n,m}$. We will investigate such problems in another paper.

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